Dimensional Primality: A Geometric Hierarchy of Integers via Discrete Hyper-Rectangles

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Abstract

We develop a geometric interpretation of integer factorization based on discrete hyper-rectangles whose side lengths are integers greater than one. We show that the maximum dimension in which an integer N can be realized as a genuine hyper-rectangle is exactly $\Omega(N)$, the number of prime factors of N counted with multiplicity. This yields a strictly nested hierarchy of integer sets $D_1 \supset D_2 \supset D_3 \supset \cdots$, where D_n consists of all integers admitting a genuine n-dimensional hyper-rectangular representation. We use this hierarchy to define a notion of *dimensional primality*: for each dimension n, the "n-dimensional primes" are precisely the integers that fail to belong to D_n, which coincide with the classical $(n-1)$-almost primes (together with 1). We further show that the smallest integer admitting a genuine n-dimensional representation is 2^n, giving a clean geometric threshold marking the onset of each new dimension.

1 Introduction

The factorization of an integer is usually presented purely arithmetically. In this note we instead view factorization geometrically: an integer N is realized as the volume (or hyper-volume) of a discrete box whose side lengths are positive integers. Restricting attention to boxes whose sides are all strictly greater than one leads to a natural notion of *dimensional capacity* for each integer, governed entirely by the arithmetic function $\Omega(N)$, the number of prime factors of N counted with multiplicity.

This perspective recovers, as a special case, the elementary fact that primes cannot be written as a product of two integers both greater than one. More generally, it shows that every integer has a well-defined "maximum dimension," and that the integers excluded from a given dimension form a hierarchy that mirrors the classical notion of k-almost primes.

2 Discrete Hyper-Rectangles and Dimensional Capacity

Definition 1 (Genuine n-dimensional hyper-rectangle)**.** *A tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{N}^n$ with $x_i > 1$ for all $i = 1, \ldots, n$ is called a* genuine n-dimensional hyper-rectangle*. Its hyper-volume is*

$$V = x_1 x_2 \cdots x_n.$$

An integer N is said to admit a genuine n-dimensional representation *if there exist $x_1, \ldots, x_n > 1$ with $x_1 x_2 \cdots x_n = N$.*

Definition 2 (Ω function)**.** *For $N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, define*

$$\Omega(N) = a_1 + a_2 + \cdots + a_k,$$

the total number of prime factors of N, counted with multiplicity. By convention $\Omega(1) = 0$.

Theorem 1 (Dimensional capacity)**.** *An integer $N \geq 2$ admits a genuine n-dimensional representation if and only if $n \leq \Omega(N)$.*

Proof. ($\Leftarrow$) Suppose $n \leq \Omega(N)$. Write $N = q_1 q_2 \cdots q_{\Omega(N)}$, where the q_i are the prime factors of N listed with multiplicity (each $q_i \geq 2$). Partition this multiset of $\Omega(N)$ primes into n nonempty groups (possible since $n \leq \Omega(N)$), and let x_i be the product of the primes in the i-th group. Then each $x_i \geq 2 > 1$ and $x_1 x_2 \cdots x_n = N$, so $(x_1, \ldots, x_n)$ is a genuine n-dimensional representation of N.

($\Rightarrow$) Suppose $(x_1, \ldots, x_n)$ is a genuine n-dimensional representation of N, so each $x_i > 1$ and $N = x_1 \cdots x_n$. Since Ω is completely additive over products,

$$\Omega(N) = \sum_{i=1}^{n} \Omega(x_i).$$

Each $x_i > 1$ is an integer, hence has at least one prime factor, so $\Omega(x_i) \geq 1$ for all i. Therefore

$$\Omega(N) = \sum_{i=1}^{n} \Omega(x_i) \geq \sum_{i=1}^{n} 1 = n.$$

□

Corollary 1 (Forced collapse)**.** *If N is presented in a candidate n-dimensional form $(x_1, \ldots, x_n)$ with $n > \Omega(N)$, then at least $n - \Omega(N)$ of the coordinates x_i must equal 1. Equivalently, the representation collapses to an effective dimension of at most $\Omega(N)$.*

Proof. If $(x_1, \ldots, x_n)$ has m coordinates equal to 1 and $n - m$ coordinates greater than 1, the same additivity argument gives $\Omega(N) \geq n - m$, i.e., $m \geq n - \Omega(N)$. □

3 The Dimensional Hierarchy

Definition 3. *For $n \geq 1$, define*

$$D_n = \{N \in \mathbb{N} : N \text{ admits a genuine } n\text{-dimensional representation}\} = \{N : \Omega(N) \geq n\}.$$

Proposition 1 (Strict nesting)**.**

$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots, \qquad \bigcap_{n \geq 1} D_n = \emptyset.$$

Moreover each inclusion $D_{n+1} \subseteq D_n$ is strict, and $D_n \setminus D_{n+1} \neq \emptyset$ for every n.

Proof. The containment $D_{n+1} \subseteq D_n$ is immediate from $\Omega(N) \geq n + 1 \implies \Omega(N) \geq n$. The intersection is empty since $\Omega(N)$ is finite for every N. Strictness follows because $2^n \in D_n \setminus D_{n+1}$ for every n (see Theorem 2 below): $\Omega(2^n) = n$, so $2^n \in D_n$ but $2^n \notin D_{n+1}$. □

4 Dimensional Primality

The classical notion of a prime is exactly the statement that D_2 excludes the primes: a prime p has $\Omega(p) = 1 < 2$, so $p \notin D_2$. This motivates the following generalization.

Definition 4 (n-dimensional primes)**.** *An integer $N \geq 1$ is called* n-dimensionally prime *if $N \notin D_n$, i.e., if N does* not *admit a genuine n-dimensional representation. Equivalently,*

$$N \text{ is } n\text{-dimensionally prime} \iff \Omega(N) < n \iff \Omega(N) \leq n-1.$$

Remark 1. *The set of n-dimensionally prime integers is exactly $\{1\} \cup \{N : \Omega(N) \leq n-1\}$, which is the union of 1 together with all k-almost primes for $k = 1, \ldots, n-1$ (an integer is a k-*almost prime *if $\Omega(N) = k$). In particular:*

- *$n = 2$: the 2-dimensional primes are $\{1\} \cup \{$ordinary primes$\}$, recovering the classical case.*
- *$n = 3$: the 3-dimensional primes additionally include all semiprimes ($\Omega = 2$), e.g. $4, 6, 9, 10, 14, 15, 21, 25, \ldots$*
- *$n = 4$: additionally includes all 3-almost primes, e.g. $8, 12, 18, 20, 27, 30, \ldots$*

As n increases, each dimension's "primes" absorb the previous dimension's smallest composites, so dimensional primality is a relative*, not absolute, notion: an integer can be composite in one dimension and prime in the next.*

Remark 2 (On terminology)**.** *Unlike ordinary primes, n-dimensionally prime numbers for $n \geq 3$ need not be multiplicatively irreducible; e.g. 6 is 3-dimensionally prime ($\Omega(6) = 2 < 3$) yet factors as 2×3. The term reflects* geometric *irreducibility at level n (failure to fill an n-dimensional interior), not arithmetic irreducibility. Where ambiguity matters, the term* n-dimensional atom *or* $(n-1)$-almost prime *may be preferred.*

5 The Smallest Composite in Each Dimension

Theorem 2 (Minimal n-dimensional composite)**.** *For every $n \geq 1$, the smallest integer in D_n is 2^n. That is,*

$$\min\{N : \Omega(N) \geq n\} = 2^n,$$

and this minimum is attained uniquely by the n-cube $(2, 2, \ldots, 2)$.

Proof. First, $\Omega(2^n) = n$, so $2^n \in D_n$.

Now suppose $N \in D_n$, so $N = x_1 x_2 \cdots x_n$ with each $x_i \geq 2$ (since $x_i > 1$ and $x_i \in \mathbb{N}$). Then

$$N = \prod_{i=1}^{n} x_i \geq \prod_{i=1}^{n} 2 = 2^n,$$

with equality if and only if $x_i = 2$ for all i. Hence 2^n is the minimum element of D_n, attained uniquely by the all-2's tuple. □

Remark 3. *Theorem 2 gives a clean geometric ladder:*

$$2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 16, \quad 2^5 = 32, \ \ldots$$

Each 2^n is the threshold integer at which dimension n first becomes geometrically realizable: it is simultaneously the largest possible "n-dimensionally prime" boundary case in the limit and the smallest member of D_n, sitting at exactly $\Omega(N) = n$.

6 Summary Table

Dimension n	D_n condition	Smallest element of D_n	Newly excluded (n-dim primes added)
1	$\Omega(N) \geq 1$	2	— (all $N \geq 2$ qualify)
2	$\Omega(N) \geq 2$	4	ordinary primes
3	$\Omega(N) \geq 3$	8	semiprimes ($\Omega = 2$)
4	$\Omega(N) \geq 4$	16	3-almost primes
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\Omega(N) \geq n$	2^n	$(n-2)$-almost primes

7 Conclusion

By interpreting integer factorization geometrically as the construction of discrete hyper-rectangles, we obtain a strict hierarchy $D_1 \supset D_2 \supset D_3 \supset \cdots$ governed entirely by $\Omega(N)$, the number of prime factors counted with multiplicity. This hierarchy gives rise to a relative notion of primality: at each dimension n, a new layer of integers (the $(n-1)$-almost primes) becomes "prime" in the sense of failing to fill the n-dimensional interior, while the smallest integer to newly qualify for dimension n is always 2^n. The classical dichotomy between primes and composites is recovered as the $n = 2$ case of this broader dimensional framework.